Available online at www.sciencedirect.com
ScienceDirect
Journal of Applied Mathematics and Mechanics 70 (2006) 611–620
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

# Shock waves in an elastic medium with cubic anisotropy☆

E.I. Sveshnikova

*Moscow, Russia*

Received 1 February 2005

**Abstract**

Low-intensity shock waves, propagating along the principal diagonal of a cube in an incompressible elastic medium possessing cubic symmetry, are considered. The form of the shock adiabatic in the phase plane of shears is obtained. Sections corresponding to non-decreasing entropy at the discontinuity and the conditions of evolutionarity of the discontinuity on it are indicated. The structure of the shock waves is investigated.


Plane one-dimensional waves in a slightly non-linear elastic medium are considered. We are particularly interested in the effect on this process of the anisotropy of the medium in the plane of the wave front (wave anisotropy) and its interaction with the effects of non-linearity.

A non-linear elastic medium, in the region where the strains are not too large, will be specified by an elastic potential (the internal energy per unit volume) in the form of an expansion in series in the strains, confining ourselves to the first principal terms, which determine the non-linearity. In the general case, wave anisotropy already occurs in the quadratic terms of the expansion, and isotropic non-linearity in the fourth-order terms. In order for the effect of anisotropy and non-linearity to have the same value in the wave propagation, it was assumed in Refs. 1–3 that the coefficient of the anisotropic term in the expansion of the elastic potential in the strain components were small. The expansion itself was taken up to the fourth degree.

The purpose of the present paper is to consider non-linear waves in such a medium when there are no terms that are quadratic in the strain, representing the anisotropy, in the expansion of the elastic potential, and the principal terms of anisotropic form are cubic. At the same time they also express the non-linearity of the medium. An elastic medium with trigonal symmetry, possessing invariance of the properties of relative rotation by an angle of 120° about a certain axis, possesses such a property. An example of such a situation is the propagation of transverse waves in an elastic medium, possessing cubic-crystal symmetry, when the direction of the wave propagation coincides with the principal diagonal of the cube.

## 1. Formulation of the problem

The elastic potential of a medium. Consider plane non-linear waves, propagating in a uniform unbounded incompressible elastic medium, possessing cubic symmetry, along the principal diagonal of the cube. There are assumed to

☆ *Prikl. Mat. Mekh.* Vol. 70, No. 4, pp. 673–683, 2006.
*E-mail address:* sveshn@mech.math.msu.su.


doi:10.1016/j.jappmathmech.2006.09.012

be no prestrains in the medium. The problem is considered in Lagrangian variables in a Cartesian system of coordinates of the initial state. The direction of the wave propagation is taken to be along the $x_3 = x$ axis, and the $x_1$ and $x_2$ axes lie in the plane of the wavefront. The strain is represented by components $\partial w_i/\partial x_j$ $(i, j = 1, 2, 3)$ of the gradient of the displacement vector $w$, of which, in plane waves propagating along the $x$ axis, the variables are $\partial w_i/\partial x = u_i(x, t)$ and the remaining ones are $\partial w_i/\partial x_\alpha = \text{const} = 0$, $\alpha = 1, 2$. A consequence of the assumption that the medium is incompressible in one-dimensional waves is that there is no change in the longitudinal component of the deformation, i.e. $u_3 = \text{const}$ and we can assume $u_3 = 0$.

For a medium with cubic symmetry, the representation of the elastic potential by an expansion in series in the strain components $\varepsilon_{ij}$ up to the third powers can be found, for example, in Ref. 4. When using the components of the gradients of the displacement $u_i$ instead of $\varepsilon_{ij}$ for waves propagating along the principal diagonal of the cube, the elastic potential of an incompressible medium must be taken in the form[5]

$$\Phi(u_i, S) = \frac{1}{2}f(u_1^2 + u_2^2) + g\left(u_1u_2^2 - \frac{1}{3}u_1^3\right) + \rho_0T_0(S - S_0) \tag{1.1}$$

Note that the assumption of incompressibility is taken here solely to simplify the description. It was shown in Ref. 5 that for a medium of this type, the change in $u_3$ can be expressed in terms of the change in $u_1$ and $u_2$. This applies similarly for $[u_3]$. As a result, the elastic potential can be represented as a function of solely two components, $u_1$ and $u_2$, and can be treated as the elastic energy of a certain equivalent incompressible medium.

Here $f$ and $g$ are the elasticity constants of the medium. It is obvious that, for very small strains (the linear approximation), the coefficient $f$ is identical with the shear modulus. The coefficient $g$ serves as the parameter of anisotropy and simultaneous of non-linearity. It has the same order of magnitude as $f$, and, by an appropriate choice of the numbering of the axes $u_i$, the coefficient $g$ can be made positive. $T_0$ is the temperature in the initial state and $S$ is the entropy per unit mass. The terms with the change in entropy $S - S_0$ must be taken into account when investigating shock waves in order to determine the direction of variation of quantities. Here, in expansion (1.1), it is sufficient to confine ourselves to the first powers of the change in entropy $S - S_0$, since it is well known from the general theory of shock waves that at a discontinuity the change in the entropy is no less than two orders of magnitude less than the change in strains. A confirmation of the correctness of this assumption is obtained below by direct calculation of $S - S_0$, which turns out to be of the third order of smallness in $u_i$, and hence the entropy-strain cross term is an order of magnitude smaller than those taken into account in the formulation of the problem and is not present in the expansion.

Without loss of generality we can further take $\rho_0 = 1$. This assumption is equivalent to using the function $\Phi/\rho_0$ everywhere henceforth instead of the elastic potential (1.1).

As can be seen from formula (1.1), the function $\Phi(u_i, S)$, which specifies the elastic potential, retains its form when the coordinate axes in the $(u_1u_2)$ plane change, namely, when $u_2$ is replaced by $-u_2$ (the mirror image in the $u_1$ axis) and when the axes are rotated as a whole by an angle of $2\pi/3$. The presence of such symmetry turns out to be useful when investigating shock waves.

## 2. Conditions on the discontinuity

The shock adiabatic. Continuous one-dimensional motions of an incompressible elastic medium in Lagrangian variables are described by the following system of hyperbolic equations[1]

$$\frac{\partial v_\alpha}{\partial t} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial u_\alpha}, \quad \frac{\partial v_\alpha}{\partial x} = \frac{\partial u_\alpha}{\partial t}, \quad \alpha = 1, 2 \tag{2.1}$$

where $v_\alpha = \partial w_\alpha/\partial t$ are the components of the velocity and $u_\alpha = \partial w_\alpha/\partial x$ are the components of the shear strains. On the discontinuity, the conditions expressing the conservation laws

$$\left[\frac{\partial \Phi}{\partial u_\alpha}\right] = W^2[u_\alpha] \tag{2.2}$$

correspond to these equations.

The square brackets denote values of the jumps in the corresponding quantities: $[a] = a^+ - a^-$, where $a^+$ and $a^-$ are the values of the parameter $a$ behind the discontinuity and in front of it respectively and $W$ is the Lagrange velocity of the discontinuity front. The energy equation on the discontinuity

$$[\Phi] = \frac{1}{2}\left[\frac{\partial\Phi}{\partial u_\alpha}\right][u_\alpha] + \left(\frac{\partial\Phi}{\partial u_\alpha}\right)^-[u_\alpha] \tag{2.3}$$

serves to calculate the change in entropy $S - S^-$ in the shock wave. For values of the parameters of state in front of the discontinuity we will henceforth use the notation $u_1^- = U_1$, $u_2^- = U_2$, $S^- = S_0$, while the parameters behind the discontinuity will not be given the superscript plus.

In a medium with the elastic potential (1.1) the conditions on the discontinuity (2.2) will be taken in the form

$$\begin{aligned} (W^2 - f)(u_1 - U_1) &= g(u_2^2 - u_1^2 + U_1^2 - U_2^2) \\ (W^2 - f)(u_2 - U_2) &= 2g(u_1 u_2 - U_1 U_2) \end{aligned} \tag{2.4}$$

Eliminating $W$ from this system, we find the relation between the specified fixed state ($u_1 = U_1$ and $u_2 = U_2$) in front of the discontinuity and the possible states behind it which satisfy the conservation laws. The set obtained in this way is the shock adiabatic and is defined by the equation

$$\begin{aligned} &\mathcal{H}(u_1, u_2, U_1, U_2) \equiv \\ &\equiv (u_1^2 - u_2^2 - U_1^2 + U_2^2)(u_2 - U_2) + 2(u_1 u_2 - U_1 U_2)(u_1 - U_1) = 0 \end{aligned} \tag{2.5}$$

Eq. (2.5) contains no other constants of the medium and depends solely on the initial state $U_1$, $U_2$ of the shear strains in front of the discontinuity.

For specified $U_1$ and $U_2$, points of three branches of the curve, shown in Fig. 1, passing through the point $A$ with coordinates $u_1 = U_1$, $u_2 = U_2$, representing the initial state, and two points symmetrical about the $u_1$ axis (the point $A'$ $(U_1, -U_2)$) and symmetrical about the origin of coordinates (the point $A_1$ $(-U_1, -U_2)$), satisfy the shock adiabatic Eq. (2.5) in the phase plane $(u_1 u_2)$. As a consequence of the symmetry of the function $\Phi(u_i)$ with respect to rotation of the axes of coordinates by an angle $2\pi/3$ and mirror reflection in the $u_1$ axis, pointed out above in Section 1, it is sufficient to consider only the situation when the point $A(U_1, U_2)$ of the initial state is situated in the $(u_1, u_2)$ plane in the first quadrant inside the angle $\pi/3$. We will therefore assume $0 < U_2 < \sqrt{3}U_1$ everywhere henceforth.

In a shock wave of infinitesimal intensity, a change in the parameters is identical with their change in Riemann waves, i.e. sections of the shock adiabatic in the neighbourhood of the initial point coincide with the elements of the integral curves of Riemann waves, which were obtained previously in Ref. 5 for system of Eq. (2.1), and the following expressions were obtained for the characteristic velocities

$$c_{1,2}^2 = f \mp 2g\sqrt{u_1^2 + u_2^2}, \quad c_1 < c_2 \tag{2.6}$$

corresponding to slow Riemann waves ($c_1$) and fast Riemann waves ($c_2$). The upper sign in formula (2.6) and below corresponds to slow waves and the lower sign to fast waves. The integral curves of the two families of Riemann waves

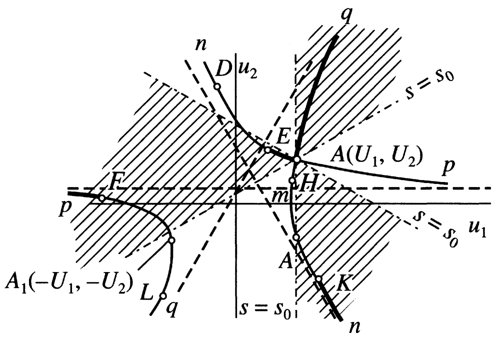


Fig. 1.

are defined by the equation

$$\frac{du_2}{du_1} = \frac{u_1 \mp \sqrt{u_1^2 + u_2^2}}{u_2}$$

At the initial point $A$ the two mutually orthogonal Riemann waves, tangential to the integral curve, define the direction of the shock adiabatic, tangential to the two branches at the initial point

$$\frac{du_2}{du_1} = \frac{U_1 \mp \sqrt{U_1^2 + U_2^2}}{U_2} \tag{2.7}$$

Hence, the shock adiabatic has a self-intersection at the initial point, which is also easy to see directly from its equation (2.5).

Curve (2.5) has three asymptotes

$$u_2 = \frac{1}{3}U_2, \quad u_2 = \pm\sqrt{3}u_1 + \frac{1}{3}(U_2 \mp U_1)$$

The first of these lies above the $u_1$ axis and is parallel to it, while the two others pass at an angle of $\pm\pi/3$ to the left of the initial point $A$. In Fig. 1 the asymptotes are represented by the dashed lines. The two points of intersection of curve (2.5) with the $u_1$ axis are situated in such a way that for one of them $0 < u_1 < U_1$ while for the other $u_1 < -2U_1$.

Hence, the shock adiabatic $\mathcal{H} = 0$ in the $(u_1 u_2)$ plane (Fig. 1) consists of three branches, departing to infinity along the asymptotes, two of them, $qAn$ and $nAp$, pass through the initial point $A$, intersecting at a right angle, while the third, $qA_1p$, passes through the point $A_1(-U_1, -U_2)$, symmetrical about the origin, and lies as a whole in the region of negative $u_1$.

## 3. The entropy degradation law at the discontinuity

In addition to the conservation laws, a consequence of which is Eq. (2.5), discontinuous solutions must satisfy the entropy degradation law $[S] = S - S_0 \geq 0$. The change in entropy is calculated using Eq. (2.3) using a specific form of (1.1) for the elastic potential $\Phi$. We have

$$2T[S] = g(u_1 - U_1)\left((u_2 - U_2)^2 - \frac{1}{3}(u_1 - U_1)^2\right) \geq 0 \tag{3.1}$$

The jump in entropy $[S]$ changes sign on the three straight lines passing through the initial point, each of which has one point of intersection with one of the branches of the shock adiabatic. The straight lines on which $[S] = 0$ are shown in Fig. 1 by the dot-dash lines, while the regions where $[S] \geq 0$ are shown hatched. Only jumps represented by sections of the shock adiabatic falling within the hatched regions can be realized.

## 4. The conditions for the evolutionarity of discontinuities

Simultaneously with the entropy degradation law, the shock waves must satisfy the conditions of evolutionarity (the necessary conditions for the stability of the discontinuity with respect to one-dimensional perturbations). These conditions consist of certain relations[2] between the velocity of the discontinuity $W$ and the characteristic velocities $c_\alpha$, specified by formulae (2.6). Two groups of inequalities of evolutionarity define two types of shock waves

$$\begin{aligned} &a)\ c_1^- \leq W \leq c_2^-, \quad W \leq c_1^+ - \text{медленные УВ} \\ &b)\ c_2^- \leq W, \quad c_1^+ \leq W \leq c_2^+ - \text{быстрые УВ} \end{aligned} \tag{4.1}$$

Here $c_\alpha^-$, $c_\alpha^+$ are the characteristic velocities in front of the discontinuity and behind it, respectively, where $\alpha = 1, 2$ and $c_1 < c_2$. To distinguish the evolution sections on the shock adiabatic, satisfying conditions (4.1), we must indicate the position of the Jouguet points on it, i.e. the points at which the velocity of the discontinuity $W$ is identical with one of the characteristic velocities $c_\alpha^-$ and $c_\alpha^+$. To do this we will obtain the velocity of the discontinuity $W$ as a function of the points on the curve (2.5).

From the relations on the discontinuity (2.4) we obtain

$$W^2 = f + \frac{g}{\rho^2}\mathcal{H}(u_1, u_2, U_1, U_2); \quad \rho^2 = (u_1 - U_1)^2 + (u_2 - U_2)^2 \tag{4.2}$$

Here we must substitute the relation between $u_1$ and $u_2$ from the shock-adiabatic Eq. (2.5). This operation is easier to carry out in polar coordiates ρ, φ with origin at the point $A$ and by measuring the angle φ from the direction that is parallel to the axis. In this system

$$u_1 - U_1 = \rho\cos\varphi, \quad u_2 - U_2 = \rho\sin\varphi$$

In polar coordinates the shock-adiabatic equation takes the form

$$\rho = 2\frac{U_1 k + U_2(1-k^2)}{(k^2-3)\sin\varphi}, \quad k = \operatorname{tg}\varphi \tag{4.3}$$

After substituting expression (4.3) into formula (4.2) we obtain the value of the square of the velocity of the discontinuity as a function of the parameter $k = tg\varphi$, which varies monotonically along the shock adiabatic $-\infty \leq k \leq \infty$. When seeking the Jouguet points (where $W = c_\alpha^{\pm}$), instead of the quantity $W^2$ we will use the auxiliary function

$$h(k) = \frac{1}{2g}(W^2 - f) = \frac{(U_1 k - U_2)(k^2+1)}{k(k^2-3)} \tag{4.4}$$

and, instead of the characteristic velocities (2.6), the corresponding expressions

$$\frac{1}{2g}(c_{1,2}^2 - f) = \mp\sqrt{u_1^2 + u_2^2}$$

The graph of the function $h(k)$ (Fig. 2) has three vertical asymptotes passing through the points $k = 0$, $k = \pm\sqrt{3}$, which correspond to three asymptotes of the shock adiabatic in the $(u_1 u_2)$ plane (Fig. 1). On each of the asymptotes the function $h$, passing through $\infty$, changes sign. It also changes sign at the only point of intersection of the graph with the $k$ axis at the point $k = U_2/U_1$, which corresponds to point $A_1$ on the shock adiabatic. We recall that, due to the symmetry for the initial state, it is sufficient to consider points at which $0 < U_2 < \sqrt{3}U_1$. Here the point $A_1$ in Fig. 2 will be situated in the region $0 < k(A_1) < \sqrt{3}$ between the two right asymptotes.

When $k \to \pm\infty$ the curve $h(k)$ reaches the horizontal asymptote $mm$, on which $h = U_1 > 0$. Consequently, $h(k) > 0$ in the regions $|k| > \sqrt{3}$. Along the asymptote $h \to +\infty$ when $|k| \to \sqrt{3} + \varepsilon$ and $h \to -\infty$ when $|k| \to \sqrt{3} - \varepsilon$ (ε is a small positive quantity). The change in the sign of $h$ at the point $A_1$ indicates that $h \to +\infty$ when $k \to 0 + \varepsilon$ and $h \to -\infty$ when $k \to 0 - \varepsilon$.

The branches of the function $h(k)$ obtained in Fig. 2 correspond to the branches of the shock adiabatic in Fig. 1, and their ends in these figures are given the same letters. The ends of the curve $h(k)$, arriving at $\infty$ along the horizontal asymptote $mm$ correspond to the point $m$ with vertical tangent on the branch $qn$ of Fig. 1.

The graph $h(k)$ intersects the horizontal asymptote $h = U_1$ twice – at the point corresponding to $0 < k < U_2/U_1$ (which is of no interest) and in the region $k > \sqrt{3}$. This denotes that when $k \to -\infty$ the curve $h(k)$ approaches the horizontal asymptote from above, and when $k \to +\infty$ it approaches it from below, having, consequently, a minimum point $H$.

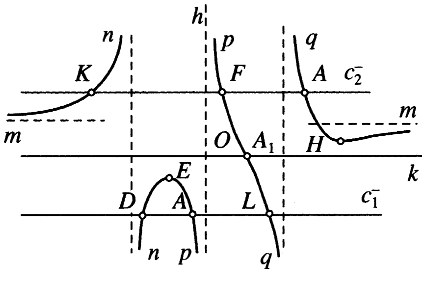


Fig. 2.

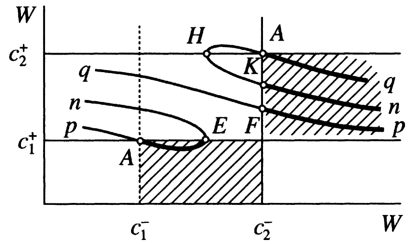


Fig. 3.

Investigation shows that the function $h(k)$ has one more extremum, which is in the region of negative $k$ and gives a maximum at the point $E$ on the branch $np$.

Two points, which are the velocities of two infinitely weak jumps, identical with the slow and fast characteristic velocities (2.6), correspond to the initial state $A(U_1, U_2)$ on the graph $h(k)$. In Fig. 2 these are the points of intersection of the curve $h(k)$ with the straight lines $h = \mp\sqrt{U_1^2 + U_2^2}$, passing below the point $E$ and above the point $H$ respectively – the extrema of the function $h(k)$ (and simultaneously of the function $W$).

There are several such points, and we must indicate which of these Jouguet points $W = c_\alpha^-$ correspond to the initial state $A$. We immediately eliminate the intersection with the branch $pq$ – this branch of the shock adiabatic does not pass through the point $A$. We will further use the fact that, in the neighbourhood of the initial point, the elements of the shock adiabatic are identical with parts of the integral curves of the Riemann waves. For the fast Riemann wave (the lower sign in formula (2.7)) we must have $k(A) > \sqrt{3}$, while for the slow wave $k(A) < 0$. On the branch $np$ there are two points where $W = c_1^-$, and the point $E$ of the maximum between them. The change in $W$ in the shock wave, beginning from the point $A$, must proceed in the direction which corresponds to breaking Riemann waves. According to the results obtained previously in Ref. 5 for the slow waves this is the direction in which $k$ decreases. This means that the initial point must be on the right of the points considered, for which $k > k(E)$. The remaining points of intersection of the horizontals considered in Fig. 2 with the graph of $h$ are the Jouguet points where $W = c_{1,2}^-$.

It is well known,[6] that the Jouguet points in the state behind the discontinuity ($W = c_{1,2}^+$) coincide with the points of the extremum of the function $W(k)$ on the shock adiabatic, or, which is the same thing in the case considered, the function $h$. The position of these points has already been obtained: the point $E$ lies on the section $An$ of the branch $pAn$, and the point $H$ lies on the section $Amn$ of the branch $qAn$ (Fig. 1).

To indicate the position of the evolution parts on the shock adiabatic, we will use the evoluationarity diagram (Fig. 3), where, on mutually orthogonal axes $W$, we plot the values of the characteristic velocities before the discontinuity $c_{1,2}^-$ and after the discontinuity $c_{1,2}^+$. The values of the velocity $W$, which fall in the hatched quadrants of this diagram, satisfy the evolutionarity conditions (4.1). In the lower quadrant, inequalities (4.1) are simultaneously satisfied in case $a$, which corresponds to slow shock waves, while in the upper quadrant inequalities (4.1) are simultaneously satisfied in case B, which corresponds to fast shock waves. At the intersection of the grid, the points $(c_1^-, c_1^+)$ and $(c_2^-, c_2^+)$ represent the initial state $A$.

In the diagram of Fig. 3 we show schematically a graph of the velocity $W$ along the shock adiabatic in accordance with Fig. 2. We can also consider this curve as the trace of the shock adiabatic on the diagram. The Jouguet points, where $W = c_{1,2}^-$, lie at the intersection with the vertical lines, and at the Jouguet points where $W = c_{1,2}^+$ the curve intersects the horizontal lines at a right angle, having extrema at these points. It can be seen from the diagram that $W = c_1^+$ at the point $E$ and $W = c_{1,2}^+$ at the point $H$. Hence, there is a single evolution section $AE$ of slow shock waves and three evolution parts $Aq$, $Kn$ and $Fp$ of fast shock waves on the shock adiabatic. In the diagram of Fig. 3 and on the shock adiabatic the evolution sections are distinguished by heavy curves and are indicated by the same letters. It turned out that all the evolution parts fell in a region where the entropy degradation law at the discontinuity (3.1) is satisfied. Hence, the conditions of evolutionarity turned out to be stronger than the entropy degradation law.

It is easy to see how the result changes if the sign of the coefficient $g$ in the expansion of the elastic potential changes from positive to a negative. The curve representing the shock adiabatic in the $(u_1u_2)$ plane in Fig. 1 remains as before, since its Eq. (2.5) does not contain $g$. When $g < 0$, the regions where $[S] \geq 0$ in the $(u_1u_2)$ plane become those shown hatched in Fig. 1. The formulae for the fast and slow characteristic velocities change their roles, and consequently, the type of Jouguet points, but not their position on the shock adiabatic. As a result, the sections $AH$ on the branch

*Amn* for the slow shock waves and the three parts, *Ap*, *Em* and *Lq*, for the shock waves will be evolution parts and simultaneously satisfy the requirement $[S] \geq 0$.

## 5. The structure of the discontinuities

We note that the evolutionarity conditions (4.1) are based on the fact that the conditions on the discontinuity consist solely of the conservation laws. Such evolutionarity is called a priori evolutionarity.[2] However, there may be other additional conditions on the discontinuity, which change the whole pattern of the situation of evolution parts on the shock adiabatic. The possibility of the occurrence of such additional conditions can only be determined when investigating the structure of the discontinuity. The appearance of discontinuous solutions is due to the hyperbolicity of the system of equations of the theory of elasticity. To investigate the structure, this model is supplemented by dissipative properties. In this paper we will take viscosity as the dissipation mechanism, and to describe the structure we will take the Kelvin-Voigt model of a viscoelastic medium. In a narrow transition zone, replacing the discontinuity, equations obtained from system (2.1) by adding viscous terms, represented by the second derivatives of the velocity[7] apply

$$\frac{\partial v_\alpha}{\partial t} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial u_\alpha} + \frac{\partial}{\partial x}\nu\frac{\partial v_\alpha}{\partial x}, \quad \frac{\partial v_\alpha}{\partial x} = \frac{\partial u_\alpha}{\partial t}, \quad \alpha = 1, 2 \tag{5.1}$$

The coefficient of viscosity $\nu$ is taken to be scalar and constant. The values of the functions $v_\alpha^\pm$, $u_\alpha^\pm$ on different sides of the discontinuity front are taken as the boundary conditions for this system when $x \to \pm\infty$. When $\nu \to 0$ system (5.1) changes into the system of equations of the theory of elasticity, while the continuous solution, if it exists, is converted into a jump.

The solution of system (5.1) in the form of a travelling wave, in which the velocity is identical with the velocity of the discontinuity $W$ being investigated, is called the stationary structure of the discontinuity

$$u_\alpha = u_\alpha(\xi), \quad v_\alpha = v_\alpha(\xi), \quad \xi = Wt - x$$

When $\xi \to \pm\infty$ this solution must reach constant values of the functions in front of and behind the discontinuity. In the new variables, i.e. in a system of coordinates moving with velocity $W$, Eq. (5.1) becomes ordinary equations with constant coefficients

$$W\frac{dv_\alpha}{d\xi} = -\frac{d}{d\xi}\frac{\partial \Phi}{\partial u_\alpha} + \frac{d}{d\xi}\nu\frac{dv_\alpha}{\partial \xi}, \quad \frac{dv_\alpha}{d\xi} = -W\frac{\partial u_\alpha}{\partial \xi}, \quad \alpha = 1, 2$$

Using the second group of equations we can eliminate $v_\alpha$ from the system, and we can integrate once with respect to $\xi$ the second-order equations obtained for the functions $u_\alpha$. We obtain

$$\nu W\frac{du_\alpha}{d\xi} = W^2 u_\alpha - \frac{\partial \Phi}{\partial u_\alpha} + A_\alpha$$

Here $A_\alpha$ are the constants of integration, which are calculated from the known values of the functions $u_\alpha$ in front of the discontinuity, where the solution must reach constant $u_\alpha = U_\alpha$ and $du_\alpha/d\xi = 0$.

For the medium considered with elastic potential $\Phi$, specified by formula (1.1), we can write the next system using the following notation introduced above in Section 4

$$(W^2 - f)/(2g) = h$$

The constants of integration $A_1$ and $A_2$ are found from the values of $u_\alpha$ and their derivatives in front of the discontinuity. We obtain

$$\frac{\nu W}{g}\frac{du_1}{d\xi} = 2hu_1 + u_1^2 - u_2^2 + A_1 \equiv L(u_1, u_2)$$

$$\frac{\nu W}{h}\frac{du_2}{d\xi} = 2hu_2 - 2u_1u_2 + A_2 \equiv M(u_1, u_2) \tag{5.2}$$

$$A_1 = -2hU_1 - U_1^2 + U_2^2, \quad A_2 = 2hU_2 + 2U_1U_2$$

The structure of the discontinuity, moving with a specified velocity $W$, exists if integral curves of the system of Eq. (5.2), connecting the state before the discontinuity $(U_1, U_2)$ with the state behind the front $(u_1^*, u_2^*)$, occur in the phase plane $(u_1u_2)$. In this case, the evolution of the solution must occur such that, when $\xi$ (i.e. the time) increases, motion along the integral curves occurs from the initial state before the front to the state behind it.

The states before and behind the discontinuity, where $du_\alpha/d\xi = 0$, for system (5.2) are stationary (singular) points. In the $(u_1u_2)$ plane these points lie at the intersection of the lines $L(u_1, u_2) = 0$ and $M(u_1, u_2) = 0$. One of the points of intersections corresponds to the state before the discontinuity and the remaining ones correspond to possible states behind the front of the jump. These points lie on the shock adiabatic (Fig. 1). The form of the curves $L = 0$ and $M = 0$, the number of points of intersection and the position in the $(u_1u_2)$ plane depend on the value of $h$, which serves as a parameter, i.e. on the velocity of the jump $W$. A graph of the change in the value of $h$ along the shock adiabatic is shown in Fig. 2. It can be seen from this graph that, depending on the chosen value of $h$ on the shock adiabatic, there can be from one to three states behind the discontinuity. Hence, when investigating the whole regions of variation of $h$ from $-\infty$ to $+\infty$ it is convenient to separate the sections in accordance with the passage of $W$ through Jouguet points.

The equation $L(u_1, u_2) = 0$ can be written in the form

$$u_2^2 = (u_1 + h)^2 - (U_1 + h)^2 + U_2^2$$

These are hyperbolae with asymptotes passing through the point $(-h, 0)$ at an angle of $\pm\,\pi/4$ to the abscissa axis. The location of the branches between the asymptotes depends on the value of $h$.

The equation $M(u_1, u_2) = 0$ also gives hyperbolae

$$u_2 = U_2(U_1 - h)/(u_1 - h)$$

with asymptotes passing through the point $(h, 0)$ parallel to the coordinate axes.

For all stationary points, their types and the behaviour of the integral curves in the neighbourhood of each of these singular points are determined, and the direction of the variation in the values of $u_1$ and $u_2$ along the integral curves with time (as $\xi$ increases) is indicated. For a structure to exist, the integral curve of Eq. (5.2), which would emerge from the stationary point, representing the state in front of the jump and which would arrive at the other stationary point as the time increases, must be found. Hence, it is clear that for this the initial singular point must possess an integral curve emerging from it.

The field of the integral curves between singular points can be described qualitatively using isoclinics. It can be seen from Eq. (5.2) that the integral curves intersect the lines $L(u_1, u_2) = 0$ on which $du_1/d\xi = 0$, with the vertical tangent, while they intersect the lines $M(u_1, u_2) = 0$ on which $du_2/d\xi = 0$, with the horizontal tangent. In this case, at the points of intersection, the integral curves have extrema, which enables us to determine the direction of variation of the quantity $u_\alpha$ along the integral curves as time increases. This procedure for constructing the field of the integral curves was carried out for all values of the parameter $h$. When $W < c_1^-$ in Fig. 3 (i.e. $h < h_A < 0$ in Fig. 2), there are four points of intersection of the curves $L = 0$ and $M = 0$. Of these the point $A$, corresponding to the state in front of the jump, is a node with incoming directions, and hence in this case the structure cannot be constructed. The three other stationary points correspond to states behind the discontinuity on the shock adiabatic, situated on the branch $Dn$, $Ap$ and $Lq$ (Figs. 1–3). There are no shock transitions in these states of the structure.

When $c_1^- < W < c_1^+$ in Fig. 3 (or when $h_A < h < h_E$ in Fig. 2), the state before the front is represented by a saddle-type singular point. One of the separatrices emerging from the point $A$ goes to the other singular point $B$ along the branch $AE$ of the shock adiabatic. The point $B$ is a node with incoming directions. These evolution discontinuities

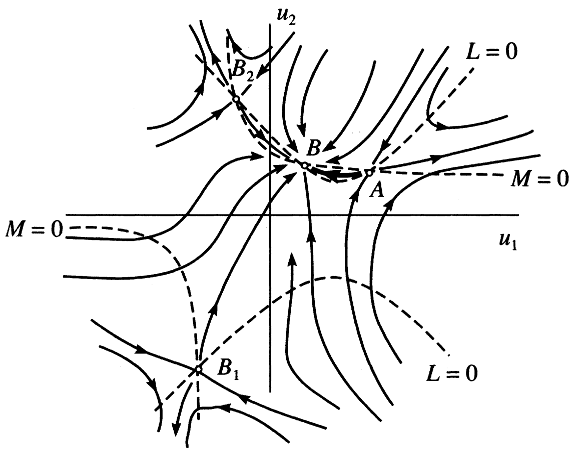


Fig. 4.

also have a structure. The scheme of the arrangement of the stationary points, the isoclinics and the integral curves for this case are shown in Fig. 4. The isoclinics are represented by the dashed curves and the integral curves are represented by the continuous curves. The arrows indicate the directions in which the functions $u_\alpha$ change with time. The integral curve representing the structure is shown by the heavy curve. The two other stationary points $B_2$ on the section $DE$ of the shock adiabatic and the point $B_3$ on the section $LH_1$ are of the saddle type. But, as can be seen from Fig. 4, the separatrice cannot enter these points from the initial state. Hence, on these sections of the shock adiabatic there is no shock-transition structure.

For $c_1^+ < W < c_2^+$ (the part on the branch $LF$ of the shock adiabatic in Figs. 1 and 2b) and for $c_2^+ < W < c_2^-$ (the sections $AH$ and $HK$ of the shock adiabatic and the section of the branch $LF$) the initial point is represented by a saddle, and the other singular points are either a node with emerging trajectories (for the section $AH$), which eliminates the possibility of constructing the structure, or these points are of the saddle type, into which the separatrices of the initial point cannot enter. These shock transitions cannot have a structure.

For $W > c_2^-$ the initial point $A$ is a node with outgoing directions. The three stationary points, representing states behind the jump, are saddles. One of the incoming separatrices for each of these three points arise from the initial point $A$. The shock transitions in the state on the evolution branches $Aq$, $Kn$ and $Fp$ of the shock adiabatic (Fig. 1) all possess a structure. The scheme of the arrangement of the integral curves for this case is shown in Fig. 5.

Hence, we have shown that all the evolution shock waves possess a stationary structure. At the same time, none of the a priori non-evolution discontinuities have a structure.

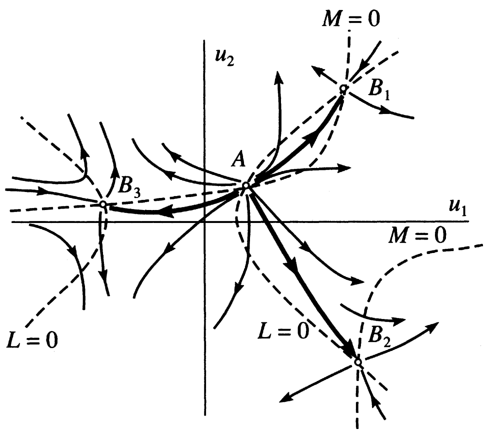


Fig. 5.

## Acknowledgement

I wish to thank A.G. Kulikovskii for his interest and for useful discussions.

This research was supported financially by the Russian Foundation for Basic Research (05-01-00219, 05-01-00375) and the Programme for the Support of the Leading Scientific Schools (NSh-4710.2006.1).

## References

1. Bland DR. *Nonlinear Dynamic Elasticity*. Toronto etc.: Waltham; 1969.
2. Kulikovskii AG, Sveshnikova EI. *Non-linear Waves in Elastic Media*. Moscow: Mosk. Litsei; 1998.
3. Kulikovskii AG, Sveshnikova EI. Investigation of the shock adiabatic of quasitransverse shock waves in a prestressed elastic medium. *Prikl Mat Mekh* 1982;**46**(5):831–40.
4. Domanski W. Asymptotic equations for weakly nonlinear elastic waves in a cubic crystal. *Intern Ser Numer Math* 1999;**129**:233–41.
5. Sveshnikova EI. Riemann waves in an elastic medium with weak cubic anisotropy. *Prikl Mat Mekh* 2005;**69**(1):75–83.
6. Kulikovskii AG. The properties of shock adiabatics in the neighbourhood of Jouguet points. *Izv Akad Nauk SSSR MZhG* 1979;**2**:184–6.
7. Kulikovskii AG, Sveshnikova EI. The structure of quasitransverse elastic shock waves. *Prikl Mat Mekh* 1987;**51**(6):926–32.

*Translated by* R.C.G.